Automated tight Lyapunov analysis for first-order methods

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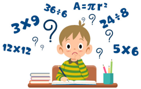
This talk

- Based on:
  - Preprint available at arXiv:2302.06713
- Content:
  - Methodology for proving algorithm convergence
  - Focus on first-order (splitting) methods for convex optimization problems

Proving convergence

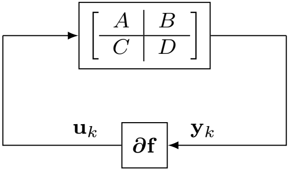
- Pages of inequalities:



- However, proofs look very similar:

- Automate!:

- Our approach:



and $$V_{k+1} \leq \rho V_k - R_k$$

One example of what we can show with our methodology

- Problem:

$$\underset{y \in \mathcal{H}}{\text{minimize}} \quad f_1(y) + f_2(y)$$

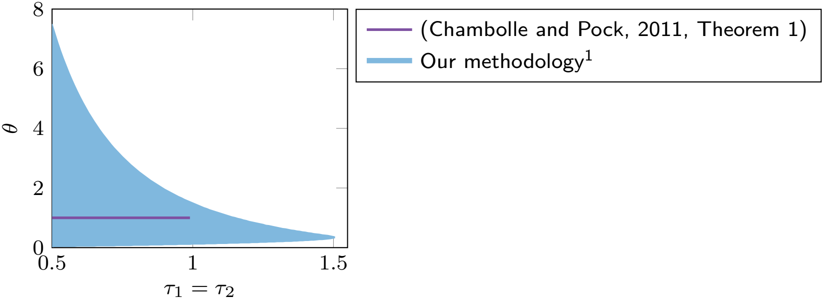
where $f_1, f_2 \in \mathcal{F}_{0,\infty}$, i.e. lower semicontinuous, proper and convex.

- Method (Chambolle and Pock, 2011, Algorithm 1):

$$\begin{aligned} x_{k+1} &= \mathrm{prox}_{\tau_1 f_1}(x_k - \tau_1 y_k), \\ y_{k+1} &= \mathrm{prox}_{\tau_2 f_2^*}(y_k + \tau_2 (x_{k+1} + \theta(x_{k+1} - x_k))) \end{aligned}$$

where $\tau_1, \tau_2 > 0$, $\theta \in \mathbb{R}$, prox is the proximal operator and f_2^* is the convex conjugate of f_2 (linear operator set to identity mapping)

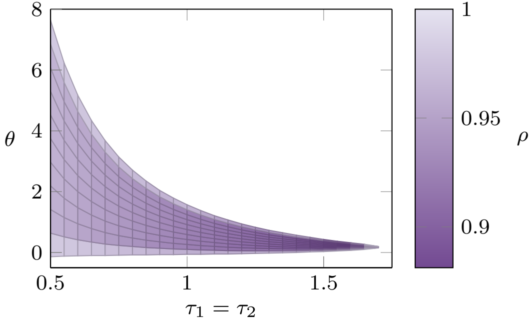
- Parameter choices that give ($\mathcal{O}(1/k)$ ergodic duality gap) convergence:



[1] Parameters evaluated on a square grid of size 0.01×0.01 with the restriction that $\tau_1 = \tau_2 \geq 0.5$

One example of what we can show with our methodology

- Let instead $f_1, f_2 \in \mathcal{F}_{0.05,50}$, i.e., 0.05-strongly convex and 50-smooth
- Parameter choices that give that the squared distance to the solution convergence ρ-linearly to zero:



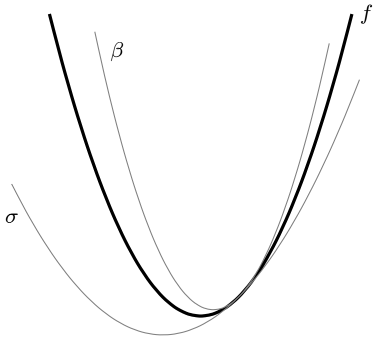
- Better rates when parameters are outside the region given in (Chambolle and Pock, 2011, Theorem 1)

Outline

Problem class - Preliminaries

- $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ real Hilbert space. Associated norm $\|\cdot\|^2 = \langle\cdot,\cdot\rangle$
- Let $0 \leq \sigma < +\infty$ and $0 \leq \beta \leq +\infty$.

  $\mathcal{F}_{\sigma,\beta}$ class of all functions $f : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ that are proper, lower semicontinuous, σ-strongly convex and β-smooth (if $\beta < +\infty$)



Problem class

- Convex optimization problem

$$\underset{y\in\mathcal{H}}{\text{minimize}} \quad \sum_{i=1}^{m} f_i(y)$$

where $f_i \in \mathcal{F}_{\sigma_i,\beta_i}$ and $0 \leq \sigma_i < \beta_i \leq +\infty$, for each $i \in [\![1, m]\!]$

- Associated inclusion problem

$$\text{find } y \in \mathcal{H} \ \text{ such that } \ 0 \in \sum_{i=1}^{m} \partial f_i(y)$$

where ∂f_i are subdifferential operators

- Problem class $\boldsymbol{\mathcal{F}_{\sigma,\beta}}$ is all $(f_1, \ldots, f_m) \in \prod_{i=1}^{m} \mathcal{F}_{\sigma_i,\beta_i}$ such that inclusion is solvable

Outline

Algorithm representation

Algorithms on state-space form[2] [3]:

$$\begin{aligned}\mathbf{x}_{k+1} &= (A \otimes \mathrm{Id})\mathbf{x}_k + (B \otimes \mathrm{Id})\mathbf{u}_k \\ \mathbf{y}_k &= (C \otimes \mathrm{Id})\mathbf{x}_k + (D \otimes \mathrm{Id})\mathbf{u}_k \\ \mathbf{u}_k &\in \boldsymbol{\partial}\mathbf{f}(\mathbf{y}_k) \\ \mathbf{F}_k &= \mathbf{f}(\mathbf{y}_k)\end{aligned}$$

where

$$A \in \mathbb{R}^{n\times n} \qquad B \in \mathbb{R}^{n\times m} \qquad C \in \mathbb{R}^{m\times n} \qquad D \in \mathbb{R}^{m\times m}$$

$$\mathbf{x}_k = \left(x_k^{(1)}, \ldots, x_k^{(n)}\right) \qquad \mathbf{y}_k = \left(y_k^{(1)}, \ldots, y_k^{(m)}\right) \qquad \mathbf{u}_k = \left(u_k^{(1)}, \ldots, u_k^{(m)}\right)$$

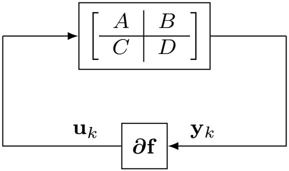
and

$$\begin{aligned}\mathbf{f} &: \mathcal{H}^m \to (\mathbb{R} \cup \{+\infty\})^m : \left(y^{(1)}, \ldots, y^{(m)}\right) \mapsto \left(f_1\left(y^{(1)}\right), \ldots, f_m\left(y^{(m)}\right)\right) \\ \boldsymbol{\partial}\mathbf{f} &: \mathcal{H}^m \to 2^{\mathcal{H}^m} : \left(y^{(1)}, \ldots, y^{(m)}\right) \mapsto \prod_{i=1}^{m} \partial f_i\left(y^{(i)}\right).\end{aligned}$$

[2] Model used in control literature, (Lessard et al., 2016), and similar to the model in (Morin et al., 2022).

[3] Let $M \in \mathbb{R}^{m\times n}$ and $\mathbf{z} = (z^{(1)}, \ldots, z^{(n)}) \in \mathcal{H}^n$. Then $(M \otimes \mathrm{Id})\mathbf{z} = \left(\sum_{j=1}^n [M]_{1,j} z^{(j)}, \ldots, \sum_{j=1}^n [M]_{m,j} z^{(j)}\right)$.

Algorithm representation



Examples:

- gradient method
- proximal point method
- proximal gradient method
- Nesterov accelerated gradient method
- gradient method with heavy-ball momentum
- triple momentum method
- FISTA
- Davis–Yin three-operator splitting method
- Chambolle–Pock method
- etc.

Algorithm representation - Chambolle–Pock method

- The problem:

$$\underset{y \in \mathcal{H}}{\text{minimize}} \quad f_1(y) + f_2(y)$$

- Method (Chambolle and Pock, 2011, Algorithm 1):

$$\begin{aligned} x_{k+1} &= \operatorname{prox}_{\tau_1 f_1}(x_k - \tau_1 y_k), \\ y_{k+1} &= \operatorname{prox}_{\tau_2 f_2^*}(y_k + \tau_2 (x_{k+1} + \theta(x_{k+1} - x_k))) \end{aligned}$$

  where $\tau_1, \tau_2 > 0$, $\theta \in \mathbb{R}$ (linear operator set to identity mapping)
- On state-space form:

$$\begin{aligned} \mathbf{x}_{k+1} &= \left(\begin{bmatrix} 1 & -\tau_1 \\ 0 & 0 \end{bmatrix} \otimes \mathrm{Id} \right) \mathbf{x}_k + \left(\begin{bmatrix} -\tau_1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \mathrm{Id} \right) \mathbf{u}_k \\ \mathbf{y}_k &= \left(\begin{bmatrix} 1 & -\tau_1 \\ 1 & \frac{1}{\tau_2} - \tau_1(1+\theta) \end{bmatrix} \otimes \mathrm{Id} \right) \mathbf{x}_k + \left(\begin{bmatrix} -\tau_1 & 0 \\ -\tau_1(1+\theta) & -\frac{1}{\tau_2} \end{bmatrix} \otimes \mathrm{Id} \right) \mathbf{u}_k \\ \mathbf{u}_k &\in \boldsymbol{\partial} \mathbf{f}(\mathbf{y}_k) \end{aligned}$$

Algorithm representation - Proximal gradient method with heavy-ball momentum

- The problem:

$$\underset{y \in \mathcal{H}}{\text{minimize}} \quad f_1(y) + f_2(y)$$

- Method:

$$x_{k+1} = \text{prox}_{\gamma f_2}(x_k - \gamma \nabla f_1(x_k) + \delta_1(x_k - x_{k-1})) + \delta_2(x_k - x_{k-1})$$

  where $\gamma > 0$ and $\delta_1, \delta_2 \in \mathbb{R}$

- On state-space form:

$$\begin{aligned}
\mathbf{x}_{k+1} &= \left(\begin{bmatrix} 1 + \delta_1 + \delta_2 & -\delta_1 - \delta_2 \\ 1 & 0 \end{bmatrix} \otimes \mathrm{Id} \right) \mathbf{x}_k + \left(\begin{bmatrix} -\gamma & -\gamma \\ 0 & 0 \end{bmatrix} \otimes \mathrm{Id} \right) \mathbf{u}_k \\
\mathbf{y}_k &= \left(\begin{bmatrix} 1 & 0 \\ 1 + \delta_1 & -\delta_1 \end{bmatrix} \otimes \mathrm{Id} \right) \mathbf{x}_k + \left(\begin{bmatrix} 0 & 0 \\ -\gamma & -\gamma \end{bmatrix} \otimes \mathrm{Id} \right) \mathbf{u}_k \\
\mathbf{u}_k &\in \boldsymbol{\partial} \mathbf{f}(\mathbf{y}_k)
\end{aligned}$$

Algorithm representation - Fixed points

- Algorithm *fixed points* $\boldsymbol{\xi}_\star = (\mathbf{x}_\star, \mathbf{u}_\star, \mathbf{y}_\star, \mathbf{F}_\star)$ satisfy

$$\begin{aligned}
\mathbf{x}_\star &= (A \otimes \mathrm{Id})\mathbf{x}_\star + (B \otimes \mathrm{Id})\mathbf{u}_\star \\
\mathbf{y}_\star &= (C \otimes \mathrm{Id})\mathbf{x}_\star + (D \otimes \mathrm{Id})\mathbf{u}_\star \\
\mathbf{u}_\star &\in \boldsymbol{\partial}\mathbf{f}(\mathbf{y}_\star) \\
\mathbf{F}_\star &= \mathbf{f}(\mathbf{y}_\star)
\end{aligned}$$

- Algorithm objective: find fixed point $\boldsymbol{\xi}_\star$, extract solution from $\boldsymbol{\xi}_\star$

Algorithm representation - Fixed-point encoding property

- We are only interested in algorithms such that

$$\text{"finding a fixed point} \quad \Longleftrightarrow \quad \text{solving inclusion problem"}$$

- More specifically[4]:
  - from each solution, it should be possible to construct a fixed point
  - from each fixed point, it should be possible to extract a solution
- Such algorithms have the *fixed-point encoding property* (FPEP)

[4] For the precise way to construct fixed points and extract solutions, see (Upadhyaya et al., 2023). This has been omitted from the presentation for clarity and simplicity

Algorithm representation - Fixed-point encoding property - Restrictions on (A, B, C, D)

- Let
$$N = \begin{bmatrix} I \\ -\mathbf{1}^\top \end{bmatrix} \in \mathbb{R}^{m\times(m-1)}$$
where $\mathbf{1}$ denotes the column vector of all ones of comfortable size
- Result:

The algorithm has the fixed-point encoding property

$$\Longleftrightarrow$$

The matrices (A, B, C, D) satisfy

$$\operatorname{ran}\begin{bmatrix} BN & 0 \\ DN & -\mathbf{1} \end{bmatrix} \subseteq \operatorname{ran}\begin{bmatrix} I - A \\ -C \end{bmatrix}$$
$$\operatorname{null}\begin{bmatrix} I - A & -B \end{bmatrix} \subseteq \operatorname{null}\begin{bmatrix} N^\top C & N^\top D \\ 0 & \mathbf{1}^\top \end{bmatrix}$$

(block row/column containing $N^\top/N$ removed when $m = 1$)

- (A, B, C, D) of all algorithms mentioned so far satisfy FPEP and is a running assumption

Algorithm representation - Well-posedness and uniqueness

- Recall: $f_i \in \mathcal{F}_{\sigma_i,\beta_i}$ for each $i \in [\![1, m]\!]$ and

$$\begin{aligned}\mathbf{x}_{k+1} &= (A \otimes \mathrm{Id})\mathbf{x}_k + (B \otimes \mathrm{Id})\mathbf{u}_k \\ \mathbf{y}_k &= (C \otimes \mathrm{Id})\mathbf{x}_k + (D \otimes \mathrm{Id})\mathbf{u}_k \\ \mathbf{u}_k &\in \boldsymbol{\partial}\mathbf{f}(\mathbf{y}_k)\end{aligned}$$

- **Well-posedness**: Can we find at least one $\mathbf{x}_{k+1}$ for each $\mathbf{x}_k$?
- **Uniqueness**: If so, is $\mathbf{x}_{k+1}$ unique?

Algorithm representation - Well-posedness and uniqueness

- Recall: $f_i \in \mathcal{F}_{\sigma_i,\beta_i}$ for each $i \in [\![1,m]\!]$ and

$$\begin{aligned}\mathbf{x}_{k+1} &= (A \otimes \mathrm{Id})\mathbf{x}_k + (B \otimes \mathrm{Id})\mathbf{u}_k \\ \mathbf{y}_k &= (C \otimes \mathrm{Id})\mathbf{x}_k + (D \otimes \mathrm{Id})\mathbf{u}_k \\ \mathbf{u}_k &\in \boldsymbol{\partial}\mathbf{f}(\mathbf{y}_k)\end{aligned}$$

- Sufficient condition for **well-posedness** and **uniqueness**:

> D *lower triangular with nonpositive diagonal and*
>
> $$\begin{aligned}I_{\text{differentiable}} &= \{i \in [\![1,m]\!] : \beta_i < +\infty\} \\ I_D &= \{i \in [\![1,m]\!] : [D]_{i,i} < 0]\}\end{aligned}$$
>
> *satisfy* $I_{\text{differentiable}} \cup I_D = [\![1,m]\!]$

- Above is a running assumption

Algorithm representation - Explicit causal implementation

- Under the sufficient condition above, the algorithm can be implemented as

$$
\begin{array}{l}
\text{for } k = 0, 1, \ldots \\
\quad \text{for } i = 1, \ldots, m \\
\quad\quad v_k^{(i)} = \sum_{j=1}^{n} [C]_{i,j} x_k^{(j)} + \sum_{j=1}^{i-1} [D]_{i,j} u_k^{(j)}, \\
\quad\quad y_k^{(i)} = \begin{cases} \operatorname{prox}_{-[D]_{i,i} f_i}\left(v_k^{(i)}\right) & \text{if } i \in I_D, \\ v_k^{(i)} & \text{if } i \notin I_D, \end{cases} \\
\quad\quad u_k^{(i)} = \begin{cases} (-[D]_{i,i})^{-1}\left(v_k^{(i)} - y_k^{(i)}\right) & \text{if } i \in I_D, \\ \nabla f_i\left(y_k^{(i)}\right) & \text{if } i \notin I_D, \end{cases} \\
\quad \mathbf{x}_{k+1} = \left(x_{k+1}^{(1)}, \ldots, x_{k+1}^{(n)}\right) = (A \otimes \mathrm{Id})\mathbf{x}_k + (B \otimes \mathrm{Id})\mathbf{u}_k,
\end{array}
$$

- Many fixed-parameter first-order methods on this form!

Outline

Lyapunov inequalities

- Let $\boldsymbol{\xi}_k = (\mathbf{x}_k, \mathbf{u}_k, \mathbf{y}_k, \mathbf{F}_k)$ and $\boldsymbol{\xi}_\star = (\mathbf{x}_\star, \mathbf{u}_\star, \mathbf{y}_\star, \mathbf{F}_\star)$
- Many first-order methods analyzed using *Lyapunov inequalities*

$$V(\boldsymbol{\xi}_{k+1}, \boldsymbol{\xi}_\star) \leq \rho V(\boldsymbol{\xi}_k, \boldsymbol{\xi}_\star) - R(\boldsymbol{\xi}_k, \boldsymbol{\xi}_\star)$$

  where $\rho \in [0, 1]$,
    - $V : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ is a *Lyapunov function*
    - $R : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ is a *residual function*

  and $\mathcal{S} = \mathcal{H}^n \times \mathcal{H}^m \times \mathcal{H}^m \times \mathbb{R}^m$
- Traditional way to find Lyapunov inequalities:
    - Use inequalities for the function classes involved (e.g. $\mathcal{F}_{\sigma_i, \beta_i}$)
    - Combine with algorithm updates
    - Manipulate to arrive at a Lyapunov inequality
- We want to automatically find such Lyapunov inequalities!

Lyapunov inequalities - Quadratic ansatzes

- We consider quadratic ansatzes of V and R:

$$V(\boldsymbol{\xi}, \boldsymbol{\xi}_\star) = \langle (\mathbf{x} - \mathbf{x}_\star, \mathbf{u}, \mathbf{u}_\star), (Q \otimes \mathrm{Id})(\mathbf{x} - \mathbf{x}_\star, \mathbf{u}, \mathbf{u}_\star) \rangle + q^\top (\mathbf{F} - \mathbf{F}_\star)$$
$$R(\boldsymbol{\xi}, \boldsymbol{\xi}_\star) = \langle (\mathbf{x} - \mathbf{x}_\star, \mathbf{u}, \mathbf{u}_\star), (S \otimes \mathrm{Id})(\mathbf{x} - \mathbf{x}_\star, \mathbf{u}, \mathbf{u}_\star) \rangle + s^\top (\mathbf{F} - \mathbf{F}_\star)$$

  where $Q, S \in \mathbb{S}^{n+2m}$, $q, s \in \mathbb{R}^m$ parameterize the functions[5]
- Our methodology searches for/provides (Q, q, S, s) that gives a valid Lyapunov inequality

[5] Inner-product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}^d$ is given by

$$\langle \mathbf{z}_1, \mathbf{z}_2 \rangle = \sum_{i=1}^{d} \left\langle z_1^{(i)}, z_2^{(i)} \right\rangle$$

for each $\mathbf{z}_i = (z_i^{(1)}, \ldots, z_i^{(d)}) \in \mathcal{H}^d$ and $i \in [\![1, 2]\!]$

Lyapunov inequalities - Lower bounds

- However, we do not know (Q, q, S, s) that parameterize V and R in advance $\implies$ can not control convergence conclusions
- Solution: enforce nonnegative quadratic lower bounds on V and R

$$V(\boldsymbol{\xi}_k, \boldsymbol{\xi}_\star) \geq \langle (\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star), (P \otimes \mathrm{Id})(\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star) \rangle + p^\top (\mathbf{F}_k - \mathbf{F}_\star) \geq 0$$
$$R(\boldsymbol{\xi}_k, \boldsymbol{\xi}_\star) \geq \langle (\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star), (T \otimes \mathrm{Id})(\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star) \rangle + t^\top (\mathbf{F}_k - \mathbf{F}_\star) \geq 0$$

where $P, T \in \mathbb{S}^{n+2m}$ and $p, t \in \mathbb{R}^m$ are fixed

Lyapunov inequalities - Lower bounds - Convergence conclusions

- Recall:
  - $V(\boldsymbol{\xi}_{k+1}, \boldsymbol{\xi}_\star) \leq \rho V(\boldsymbol{\xi}_k, \boldsymbol{\xi}_\star) - R(\boldsymbol{\xi}_k, \boldsymbol{\xi}_\star)$
  - $V(\boldsymbol{\xi}_k, \boldsymbol{\xi}_\star) \geq \langle (\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star), (P \otimes \mathrm{Id})(\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star) \rangle + p^\top (\mathbf{F}_k - \mathbf{F}_\star) \geq 0$
  - $R(\boldsymbol{\xi}_k, \boldsymbol{\xi}_\star) \geq \langle (\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star), (T \otimes \mathrm{Id})(\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star) \rangle + t^\top (\mathbf{F}_k - \mathbf{F}_\star) \geq 0$
- For $\rho \in [0, 1[$:

$$0 \leq \langle (\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star), (P \otimes \mathrm{Id})(\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star) \rangle + p^\top (\mathbf{F}_k - \mathbf{F}_\star) \leq \rho^k V(\boldsymbol{\xi}_0, \boldsymbol{\xi}_\star) \to 0$$

  i.e., lower bound

$$\Big\{ \langle (\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star), (P \otimes \mathrm{Id})(\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star) \rangle + p^\top (\mathbf{F}_k - \mathbf{F}_\star) \Big\}_{k \in \mathbb{N}_0}$$

  converges ρ-linearly to 0
- For $\rho = 1$, a telescoping summation gives

$$0 \leq \sum_{k=0}^{\infty} \Big(\langle (\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star), (T \otimes \mathrm{Id})(\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star) \rangle + t^\top (\mathbf{F}_k - \mathbf{F}_\star) \Big) \leq V(\boldsymbol{\xi}_0, \boldsymbol{\xi}_\star)$$

  i.e., lower bound

$$\Big\{ \langle (\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star), (T \otimes \mathrm{Id})(\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star) \rangle + t^\top (\mathbf{F}_k - \mathbf{F}_\star) \Big\}_{k \in \mathbb{N}_0}$$

  is summable (and converges to zero)

Lyapunov inequalities - Full definition

- (P, p, T, t, ρ)-*quadratic Lyapunov inequality* for algorithm and $\boldsymbol{\mathcal{F}_{\sigma,\beta}}$:
  C1 $V(\boldsymbol{\xi}_{k+1}, \boldsymbol{\xi}_\star) \leq \rho V(\boldsymbol{\xi}_k, \boldsymbol{\xi}_\star) - R(\boldsymbol{\xi}_k, \boldsymbol{\xi}_\star)$
  C2 $V(\boldsymbol{\xi}_k, \boldsymbol{\xi}_\star) \geq \langle (\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star), (P \otimes \mathrm{Id})(\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star) \rangle + p^\top (\mathbf{F}_k - \mathbf{F}_\star) \geq 0$
  C3 $R(\boldsymbol{\xi}_k, \boldsymbol{\xi}_\star) \geq \langle (\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star), (T \otimes \mathrm{Id})(\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star) \rangle + t^\top (\mathbf{F}_k - \mathbf{F}_\star) \geq 0$
- Technical difficulty: We only want this to hold for algorithm-consistent points $\boldsymbol{\xi}_k$, fixed points $\boldsymbol{\xi}_\star$, and $\mathbf{f} \in \boldsymbol{\mathcal{F}_{\sigma,\beta}}$

Outline

Main result

Given:

- Problem class $\mathcal{F}_{\sigma,\beta}$
- A first-order method on state-space form, i.e., (A, B, C, D)
- (P, p, T, t, ρ) deciding convergence conclusions

We provide:

- A necessary and sufficient condition for the existence of a (P, p, T, t, ρ)-quadratic Lyapunov inequality
- Parameters (Q, q, S, s) of V and R if one exists

Main result - Necessary and sufficient condition

There exists a (P, p, T, t, ρ)-quadratic Lyapunov inequality

if and only if[6]

a particular SDP involving (Q, q, S, s) is feasible

$$\mathrm{C1}\begin{cases}\lambda^{\mathrm{C1}}_{(l,i,j)} \geq 0 \text{ for each } l \in [\![1,m]\!] \text{ and distinct } i,j \in \{\varnothing,+,\star\},\\ \Sigma_\varnothing^\top(\rho Q - S)\Sigma_\varnothing - \Sigma_+^\top Q \Sigma_+ + \sum_{l=1}^{m} \sum_{\substack{i,j\in\{\varnothing,+,\star\}\\ i\neq j}} \lambda^{\mathrm{C1}}_{(l,i,j)} \mathbf{M}_{(l,i,j)} \succeq 0,\\ \begin{bmatrix}\rho q - s\\ -q\end{bmatrix} + \sum_{l=1}^{m} \sum_{\substack{i,j\in\{\varnothing,+,\star\}\\ i\neq j}} \lambda^{\mathrm{C1}}_{(l,i,j)} \mathbf{a}_{(l,i,j)} = 0,\end{cases}$$

$$\mathrm{C2}\begin{cases}\lambda^{\mathrm{C2}}_{(l,i,j)} \geq 0 \text{ for each } l \in [\![1,m]\!] \text{ and distinct } i,j \in \{\varnothing,\star\},\\ \Sigma_\varnothing^\top(Q - P)\Sigma_\varnothing + \sum_{l=1}^{m} \sum_{\substack{i,j\in\{\varnothing,\star\}\\ i\neq j}} \lambda^{\mathrm{C2}}_{(l,i,j)} \mathbf{M}_{(l,i,j)} \succeq 0,\\ \begin{bmatrix}q - p\\ 0\end{bmatrix} + \sum_{l=1}^{m} \sum_{\substack{i,j\in\{\varnothing,\star\}\\ i\neq j}} \lambda^{\mathrm{C2}}_{(l,i,j)} \mathbf{a}_{(l,i,j)} = 0,\end{cases}$$

$$\mathrm{C3}\begin{cases}\lambda^{\mathrm{C3}}_{(l,i,j)} \geq 0 \text{ for each } l \in [\![1,m]\!] \text{ and distinct } i,j \in \{\varnothing,\star\},\\ \Sigma_\varnothing^\top(S - T)\Sigma_\varnothing + \sum_{l=1}^{m} \sum_{\substack{i,j\in\{\varnothing,\star\}\\ i\neq j}} \lambda^{\mathrm{C3}}_{(l,i,j)} \mathbf{M}_{(l,i,j)} \succeq 0,\\ \begin{bmatrix}s - t\\ 0\end{bmatrix} + \sum_{l=1}^{m} \sum_{\substack{i,j\in\{\varnothing,\star\}\\ i\neq j}} \lambda^{\mathrm{C3}}_{(l,i,j)} \mathbf{a}_{(l,i,j)} = 0,\end{cases}$$

[6] Assuming dimension independence and Slater condition

Main result - How did we find this condition?

- Let us look at **C1**:[7] $V(\boldsymbol{\xi}_+, \boldsymbol{\xi}_\star) \le \rho V(\boldsymbol{\xi}, \boldsymbol{\xi}_\star) - R(\boldsymbol{\xi}, \boldsymbol{\xi}_\star)$
- **C1** equivalent to that optimal value of

$$\begin{array}{ll} \text{maximize} & V(\boldsymbol{\xi}_+, \boldsymbol{\xi}_\star) - \rho V(\boldsymbol{\xi}, \boldsymbol{\xi}_\star) + R(\boldsymbol{\xi}, \boldsymbol{\xi}_\star) \\ \text{subject to} & \boldsymbol{\xi} \text{ is algorithm consistent for } \mathbf{f}, \\ & \boldsymbol{\xi}_+ \text{ is a successor of } \boldsymbol{\xi} \text{ for } \mathbf{f}, \\ & \boldsymbol{\xi}_\star \text{ is a fixed point for } \mathbf{f}, \\ & \mathbf{f} \in \boldsymbol{\mathcal{F}}_{\boldsymbol{\sigma}, \boldsymbol{\beta}}, \end{array} \tag{PEP}$$

  is nonpositive!
- Arrived at the condition using:
  - Convex interpolation conditions (Taylor et al., 2017*b*)
  - Performance estimation problem (PEP) reformulations (Drori and Teboulle, 2014)

[7] We use the same trick for **C2** and **C3**

Outline

Numerical results - Douglas–Rachford method

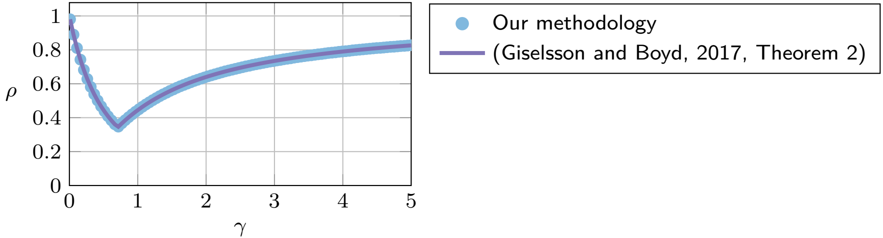
- The problem:

$$\underset{y\in\mathcal{H}}{\text{minimize}} \quad f_1(y) + f_2(y)$$

  where $f_1 \in \mathcal{F}_{1,2}$ and $f_2 \in \mathcal{F}_{0,\infty}$
- Douglas–Rachford method:

$$\begin{aligned} y_k^{(1)} &= \operatorname{prox}_{\gamma f_1}(x_k) \\ y_k^{(2)} &= \operatorname{prox}_{\gamma f_2}\left(2y_k^{(1)} - x_k\right) \\ x_{k+1} &= x_k + \lambda\left(y_k^{(2)} - y_k^{(1)}\right) \end{aligned}$$

  where $\gamma \in \mathbb{R}_{++}$ and $\lambda \in \mathbb{R} \setminus \{0\}$ ($\lambda = 1$ in the plot below)
- $(P, p, T, t, \rho) \implies$ squared distance to the solution convergence ρ-linearly to zero[8]



[8]Smallest ρ via bisection search

Numerical results - Gradient method with heavy-ball momentum

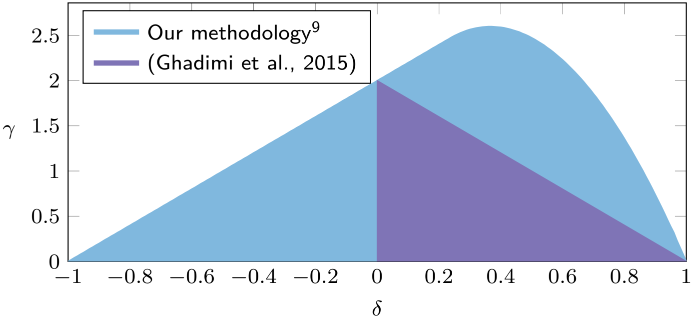
- The problem:

$$\underset{y\in\mathcal{H}}{\text{minimize}} \quad f_1(y)$$

  where $f_1 \in \mathcal{F}_{0,1}$
- Gradient method with heavy-ball momentum:

$$x_{k+1} = x_k - \gamma\nabla f_1(x_k) + \delta(x_k - x_{k-1})$$

- $(P, p, T, t, \rho) \implies \lim_{k\to\infty}(f_1(x_k) - f_1(x_\star)) = 0$ and $f_1\left(\frac{1}{K}\sum_{k=1}^{K} x_k\right) - f_1(x_\star) = \mathcal{O}\left(\frac{1}{K}\right)$



[9] Parameters evaluated on a square grid of size 0.01×0.01

Outline

Summary and outlook

- **Summary:**
  - A framework for automated convergence proofs for first-order methods used to solve convex optimization problems
  - Introduced a state-space representation based on matrices A, B, C, D
  - Introduced a necessary and sufficient condition for the existence of quadratic Lyapunov inequalities
  - Numerical examples extending previous results
- **Outlook:**
  - Change $f_i \in \mathcal{F}_{\sigma_i, \beta_i}$ to any function class that has quadratic interpolation constraints:
    - class of smooth functions (Taylor et al., 2017*a*)
    - class of convex and quadratically upper bounded functions (Goujaud et al., 2022)
    - class of convex and Lipschitz continuous functions (Taylor et al., 2017*a*)
    - class of smooth hypoconvex (weakly convex) functions (Rotaru et al., 2022)
    - class of smooth functions satisfying the Polyak-Łojasiewicz inequality (Abbaszadehpeivasti et al., 2022)
  - Extend algorithm representation to allow for more types of oracles:
    - Frank–Wolfe-type oracles (Taylor et al., 2017*a*)
    - Bregman-type oracles (Dragomir et al., 2022)
    - approximate proximal operator oracles (Barré et al., 2022)
  - Allow multiple evaluations of the same subdifferential ∂f_i during the same iteration
    - enabling analysis of, e.g., the forward–backward–forward splitting method of Tseng (Tseng, 2000)
  - Extend the quadratic Lyapunov function and the quadratic residual function ansatzes to not only contain the current iterate $\boldsymbol{\xi}_k$, but some history $\boldsymbol{\xi}_k, \boldsymbol{\xi}_{k-1}, \ldots, \boldsymbol{\xi}_{k+1-h}$ for some integer $h \geq 1$
  - Use methodology to find computer-aided proofs of analytical Lyapunov inequalities and convergence results

Thank you

arXiv (with code link)

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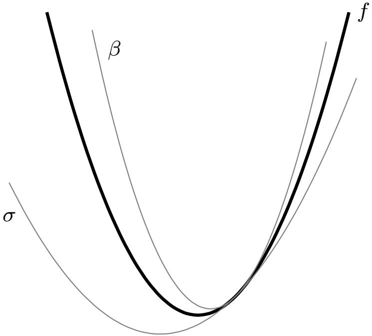
Appendix - Preliminaries

- $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ real Hilbert space. Associated norm $\|\cdot\|^2 = \langle\cdot,\cdot\rangle$
- Let $f : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$. Then:
  - (i) *effective domain* of f is the set $\operatorname{dom} f = \{x \in \mathcal{H} \mid f(x) < +\infty\}$
  - (ii) f *proper* if $\operatorname{dom} f \neq \emptyset$
  - (iii) *subdifferential* of a proper function f is the set-valued operator $\partial f : \mathcal{H} \to 2^{\mathcal{H}}$ such that $x \mapsto \{u \in \mathcal{H} \mid \forall y \in \mathcal{H},\ f(y) \geq f(x) + \langle u, y - x\rangle\}$
- Let $f\colon \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ and $\sigma, \beta \in \mathbb{R}_+$. The function f is:
  - (i) *convex* if $f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y)$ for each $x, y \in \mathcal{H}$ and $0 \leq \lambda \leq 1$
  - (ii) *σ-strongly convex* if f is proper and $f - (\sigma/2)\|\cdot\|^2$ is convex
  - (iii) *β-smooth* if f is differentiable and $\|\nabla f(x) - \nabla f(y)\| \leq \beta \|x - y\|$ for each $x, y \in \mathcal{H}$

Appendix - More preliminaries

- Let $0 \leq \sigma < +\infty$ and $0 \leq \beta \leq +\infty$. $\mathcal{F}_{\sigma,\beta}$ class of all functions $f : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ that are proper, lower semicontinuous, σ-strongly convex and β-smooth (if $\beta < +\infty$)



- Let $f \in \mathcal{F}_{0,\infty}$ and $\gamma > 0$. Then the *proximal operator* $\operatorname{prox}_{\gamma f} : \mathcal{H} \to \mathcal{H}$ is defined as the single-valued operator given by

$$\operatorname{prox}_{\gamma f}(x) = \underset{z \in \mathcal{H}}{\operatorname{argmin}} \left(f(z) + \frac{1}{2\gamma} \|x - z\|^2 \right)$$

  for each $x \in \mathcal{H}$

- The *convex conjugate* of f, denoted $f^* : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$, is the proper, lower semicontinuous and convex function given by $f^*(u) = \sup_{x \in \mathcal{H}} (\langle u, x \rangle - f(x))$ for each $u \in \mathcal{H}$

Appendix - Some choices of (P, p, T, t, ρ)

Suppose $\rho \in [0, 1[$, let e_i be ith standard basis vector and

$$(P, p, T, t) = \left(\begin{bmatrix} C & D & -D \end{bmatrix}^\top e_i e_i^\top \begin{bmatrix} C & D & -D \end{bmatrix}, 0, 0, 0 \right).$$

Then

$$\langle (\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star), (P \otimes \mathrm{Id})(\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star) \rangle + p^\top (\mathbf{F}_k - \mathbf{F}_\star) = \left\| y_k^{(i)} - y_\star \right\|^2 \geq 0$$

and the distance to the solution squared converges ρ-linear to zero.

Appendix - Some choices of (P, p, T, t, ρ)

Suppose $\rho = 1$, $m = 1$ and let

$$(P, p, T, t) = (0, 0, 0, 1).$$

Then

$$\langle (\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star), (T \otimes \mathrm{Id})(\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star)\rangle + t^\top(\mathbf{F}_k - \mathbf{F}_\star) = f_1\left(y_k^{(1)}\right) - f_1(y_\star) \geq 0$$

which gives

- function value suboptimality converges to zero
- $\mathcal{O}(1/k)$ ergodic function value suboptimality convergence (via Jensen's inequality)

Appendix - Some choices of (P, p, T, t, ρ)

Suppose $\rho = 1$ and let

$$(P, p, T, t) = \left(0, 0, \begin{bmatrix} C & D & -D \\ 0 & 0 & I \end{bmatrix}^\top \begin{bmatrix} 0 & -\frac{1}{2}I \\ -\frac{1}{2}I & 0 \end{bmatrix} \begin{bmatrix} C & D & -D \\ 0 & 0 & I \end{bmatrix}, \mathbf{1}\right).$$

Then

$$\begin{aligned} &\langle (\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star), (T \otimes \mathrm{Id})(\mathbf{x}_k - \mathbf{x}_\star, \mathbf{u}_k, \mathbf{u}_\star) \rangle + t^\top (\mathbf{F}_k - \mathbf{F}_\star) \\ &= \sum_{i=1}^{m} \left(f_i\left(y_k^{(i)}\right) - f_i\left(y_\star^{(i)}\right) - \left\langle u_\star^{(i)}, y_k^{(i)} - y_\star^{(i)} \right\rangle \right) \\ &= \mathcal{L}(\mathbf{y}_k, \mathbf{u}_\star) - \mathcal{L}(\mathbf{y}_\star, \mathbf{u}_k) \geq 0 \end{aligned}$$

where $\mathcal{L} : \mathcal{H}^m \times \mathcal{H}^m \to \mathbb{R}$ is a *Lagrangian function* giving

- duality gap converges to zero,
- $\mathcal{O}(1/k)$ ergodic duality gap convergence (via Jensen's inequality).

Reduces to function value suboptimality when $m = 1$.